Roche Lobes and the Morphologies of Close Binary Stars

Dan Bruton, Ph.D., SFA Observatory, May 8, 2004

If we assume that the stars of a binary star system are in a state of hydrostatic equilibrium, then we can develop a method of obtaining the approximate shape of the stars with some fundamental physics. Our goal is to find the radius, \( r \), as a function of the angle from the pole, \( \theta \), and azimuthal angle, \( \phi \). Consider two stars separated by a distance \( R \) that rotate around one another with an angular velocity of \( \omega \) about an axis that is parallel to the \( z \)-axis.

The shape of the stars can be determined by looking at the forces on a small element of mass, \( m \), within either star.

\[
\sum \vec{F} = m\vec{a}
\]

\[
\vec{F}_{\text{pressure}} + \vec{F}_{\text{gravity}} = -m\omega^2 \rho'
\]

where \( \rho' \) is the axial distance from the axis of rotation. The prime (') is used here for coordinates relative to the axis of rotation and terms that do not have primes will be used for coordinates relative to the center of \( m_1 \). The first term above is the pressure force (or pressure-gradient force) and is defined to be the force due to differences of pressure within a fluid mass. The pressure force per unit volume is equal to the negative gradient of the pressure. This gives us

\[
\vec{F}_{\text{pressure}} = -\nabla P = -\frac{m}{\rho_m} \nabla P
\]

where \( \rho_m \) is the mass density of the volume \( V \) of mass \( m \). Note that the rho here is density and should not be confused with the rho vector above.
Notice that the vector sum of two of the terms in Equation (1) can be expressed as the negative gradient of a potential $\Psi$.

$$\vec{F}_{\text{gravity}} + m\omega^2 \vec{\rho} = -\frac{Gm_1}{r_1^2} \hat{r}_1 - \frac{Gm_2}{r_2^2} \hat{r}_2 + m\omega^2 \vec{\rho} = -m \nabla \Psi$$  \hspace{1cm} (3)

where $r_1$ and $r_2$ are the distances of small element of mass, $m$, from the centers of $m_1$ and $m_2$ respectively. From Equations (1), (2) and (3) we see that

$$\nabla P = -\rho_m \nabla \Psi$$ \hspace{1cm} (4)

which implies that $\nabla P$ and $\nabla \Psi$ are parallel. If we take the curl of Equation (4) we find

$$\nabla \times \nabla P = \nabla \times (-\rho_m \nabla \Psi) = -\rho_m \nabla \times \nabla \Psi - \nabla \rho_m \times \nabla \Psi.$$

The curl of gradient of any function is zero. Therefore,

$$\nabla \rho_m \times \nabla \Psi = 0$$

from which it follows that $\nabla \rho_m$ is parallel to both $\nabla P$ and $\nabla \Psi$. This circumstance implies that surfaces of constant density, pressure and $\Psi$ coincide. So if we find shapes corresponding to constant $\Psi$, then we have found the possible shapes of the stars of a binary system. From equation (3) we find that

$$\Psi = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} - \frac{1}{2} \omega^2 \rho^2.$$ \hspace{1cm} (5)

The coordinates of the small element of mass, $m$, can be expressed in Cartesian, spherical, or cylindrical coordinate systems.

$$\begin{align*}
    x &= r \sin \theta \cos \phi \\
    y &= r \sin \theta \sin \phi \\
    z &= r \cos \theta \\
    \rho &= r \sin \theta = \sqrt{x^2 + y^2}
\end{align*}$$ \hspace{1cm} (6)

Notice that the center of mass lies on the x-axis and its distance from $m_1$ is

$$X \equiv \frac{\sum m_i x_i}{\sum m_i} = \frac{m_2 R}{m_1 + m_2} = \frac{qR}{1 + q}$$ \hspace{1cm} (7)

where $q = m_2/m_1$. 
Now we can find $r_1$, $r_2$ and $\rho'$ in terms of the coordinates relative to our origin in the center of $m_1$.

\[
\begin{align*}
    r_1 &= r \\
    r_2 &= \sqrt{(x - R)^2 + y^2 + z^2} = \sqrt{r^2 - 2xR + R^2} \\
    \rho' &= \sqrt{(x - \bar{x})^2 + y^2} = \sqrt{\rho^2 - 2\bar{x}x + \bar{x}^2}
\end{align*}
\] (8)

Using Equations (5), (6), and (8), we get

\[
\Psi(r, \theta, \phi) = -\frac{Gm_1}{r} - \frac{Gm_2}{\sqrt{r^2 - 2rR \sin \theta \cos \phi + R^2}} - \frac{1}{2} \omega^2 \left(r^2 \sin^2 \theta - 2 \bar{x} \sin \theta \cos \phi + \bar{x}^2 \right).
\] (9)

From Kepler’s third law we have

\[
\omega^2 = \frac{G(m_1 + m_2)}{R^3}.
\] (10)

Further, let $\lambda = \sin \theta \cos \phi$, $\mu = \sin \theta \sin \phi$, and $\nu = \cos \theta$. Using Equations (9), (7), and (10), we get

\[
\begin{align*}
    \Psi(r, \lambda, \nu) &= -\frac{Gm_1}{r} - \frac{Gm_2}{\sqrt{r^2 - 2rR\lambda + R^2}} - \frac{1}{2} \frac{G(m_1 + m_2)}{R^3} \left(r^2 (1 - \nu^2) - 2 \frac{qR}{1+q} \frac{r\lambda}{1+q} + \left(\frac{qR}{1+q}\right)^2 \right) \\
    \Psi(r, \lambda, \nu) &= -\frac{Gm_1}{R} \left[\frac{R}{r} + \frac{Rq}{\sqrt{r^2 - 2rR\lambda + R^2}} + \frac{1}{2} \left(1 + \frac{q}{R} \right) \left(r^2 (1 - \nu^2) - 2 \frac{qR}{1+q} \frac{r\lambda}{1+q} + \left(\frac{qR}{1+q}\right)^2 \right) \right] \\
    \Psi(r, \lambda, \nu) &= -\frac{Gm_1}{R} \left[\frac{R}{r} + \frac{Rq}{\sqrt{r^2 - 2rR\lambda + R^2}} + \frac{1}{2} \left(1 + \frac{q}{R} \right) r^2 (1 - \nu^2) - \frac{q}{R} r\lambda + \frac{1}{2} \frac{q^2}{1+q} \right] \\
    \Psi(r, \lambda, \nu) &= -\frac{Gm_1}{R} \left[\frac{R}{r} + \frac{Rq}{\sqrt{r^2 - 2rR\lambda + R^2}} - \frac{r\lambda}{R} + \frac{1}{2} \left(1 + \frac{q}{R} \right) r^2 (1 - \nu^2) + \frac{1}{2} \frac{q^2}{1+q} \right]
\end{align*}
\] (11)

We can now define a new potential function as Kopal did in 1959.

\[
\Omega = -\frac{R\Psi}{Gm_1} - \frac{1}{2} \frac{q^2}{1+q}
\] (12)

We also notice that $r$ can be expressed as a fraction of $R$ in the unitless term

\[
\bar{r} = \frac{r}{R}
\] (13)
Using Equations (11), (12), and (13) we get

\[ \Omega = \frac{1}{r} + q \left( \frac{1}{\sqrt{r^2 - 2r\lambda + 1}} - \frac{1}{r} \right) + \frac{1}{2} (1 + q)(1 - \nu^2) \bar{r}^2 \]  \quad (14)

At the pole of \( m_1 \), \( \theta = 0^\circ \), \( \lambda = 0 \), \( \nu = 1 \), and

\[ \Omega = \frac{1}{\bar{r}_{\text{pole}}} + q \left( \frac{1}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} \right) = \text{constant} \, . \]  \quad (15)

So to find the surface of the star we only need to find \( r \) for each \( \theta \) and \( \phi \) using

\[ \frac{1}{\bar{r}_{\text{pole}}} + q \left( \frac{1}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} \right) = \frac{1}{\bar{r}} + q \left( \frac{1}{\sqrt{\bar{r}^2 - 2\bar{r}\lambda + 1}} - \frac{1}{\bar{r}} \right) + \frac{1}{2} (1 + q)(1 - \nu^2) \bar{r}^2 \]  \quad (16)

or more simply

\[ \frac{1}{\bar{r}} = \frac{1}{\bar{r}_{\text{pole}}} + q \left( \frac{1}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} \right) - q \left( \frac{1}{\sqrt{\bar{r}^2 - 2\bar{r}\lambda + 1}} - \frac{1}{\bar{r}} \right) - \frac{1}{2} (1 + q)(1 - \nu^2) \bar{r}^2 \, . \]  \quad (17)

There is clearly not a simple analytical solution of \( \bar{r} \) as a function of \( \theta \) and \( \phi \). There are however, ways that we can obtain the values of \( \bar{r} \) to an arbitrary accuracy.

**Method 1**: One way is to use \( \bar{r}_{\text{pole}} \) as an initial guess for \( \bar{r} \) on the right hand side of Equation (17) and compute \( \bar{r} \) on the left hand side. The new computed value of \( \bar{r} \) can then be used on the right hand side as a better guess. Each iteration step will yield a more accurate value for \( \bar{r} \). Only a few iterations are needed since the differences between the guess and the result decrease rapidly as we will see in the example below.

**Method 2**: Another way to find \( r \) as a function of \( \theta \) and \( \phi \) is to use the Newton-Raphson method for finding the roots of a polynomial. Using Equation (17) the polynomial can be written as

\[ f(r) = \frac{\bar{r}}{\bar{r}_{\text{pole}}} + q \left( \frac{\bar{r}}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} \right) - q \left( \frac{\bar{r}}{\sqrt{\bar{r}^2 - 2\bar{r}\lambda + 1}} - \frac{\bar{r}^2}{\bar{r}} \right) - \frac{1}{2} (1 + q)(1 - \nu^2) \bar{r}^3 - 1 = 0 \, . \]  \quad (18)

With a good initial choice of the root's position, the Newton-Raphson algorithm can be applied iteratively to obtain an improved value \( \bar{r}_{n+1} \) from \( \bar{r}_n \) as follows

\[ \bar{r}_{n+1} = \bar{r}_n - \frac{f(\bar{r}_n)}{f'(\bar{r}_n)} \]

where
\[ f'(r) = \frac{1}{\tilde{r}_{\text{pole}}} + q \left( \frac{1}{\sqrt{\tilde{r}_{\text{pole}}^2 + 1}} \right) - q \left( \frac{1}{\sqrt{\tilde{r}^2 - 2\tilde{r}\lambda + 1}} - \tilde{r}(\tilde{r} - \lambda)(\tilde{r}^2 - 2\tilde{r}\lambda + 1) - 2\tilde{r}\lambda \right) - \frac{3}{2} (1 + q)(1 - \nu^2)\tilde{r}^2 \]

which gives us

\[ r_{n+1} = r_n - \frac{1}{\tilde{r}_{\text{pole}}} + q \left( \frac{1}{\sqrt{\tilde{r}_{\text{pole}}^2 + 1}} \right) - q \left( \frac{1}{\sqrt{\tilde{r}_{\text{pole}}^2 - 2\tilde{r}\lambda + 1}} - \tilde{r}_{\text{pole}}(\tilde{r}_{\text{pole}} - \lambda)(\tilde{r}_{\text{pole}}^2 - 2\tilde{r}_{\text{pole}}\lambda + 1) - 2\tilde{r}_{\text{pole}}\lambda \right) - \frac{3}{2} (1 + q)(1 - \nu^2)\tilde{r}_{\text{pole}}^3 - 1 \]

**Example:** Consider the close binary star system DM Delphinius that has a mass ratio of \( q = 0.260 \). Assume that the pole radius of the primary star is 0.467 times the separation between the two stars (\( \tilde{r}_{\text{pole}} \)). We will compute the radius of the primary star at the point on its equator closest to the secondary star (\( \tilde{r}_{\text{poin}} \)) using the two methods described above.

The results in the table to the right show that both methods eventually converge to the same value. Simple iteration appears to be the better of the two methods because it has about the same convergence rate as the Newton-Raphson method and it is clearly easier to implement in computer programs.

<table>
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<th>Method 1</th>
<th>Error</th>
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The Roche lobe is the region of space around a star in a binary system within which orbiting material is gravitationally bound to that star. If the star expands past its Roche lobe, then the material outside of the lobe will fall into the other star. It is approximately tear-drop shaped, with the apex of the tear-drop pointing towards the other star (and the apex is at the Lagrange L1 point of the system).

Close to each stellar center the equipotential surfaces are approximately spherical and concentric with the nearer star. Far from the stellar system, the equipotentials are approximately ellipsoidal and elongated parallel to the axis joining the stellar centers. A critical equipotential intersects itself at the center of mass of the system. It is this equipotential which defines the Roche lobes. The plot below shows the computed morphology of DM Delphinus, a close binary star system that almost completely fills its Roche lobes.

The two body system has five equilibrium points where a test particle can have zero velocity and zero acceleration in the frame which corotates with the primary masses about their common center of mass (Danby 1988). Three of these points lie along the line through the two primary masses and are unstable. The other two, L4 and L5, lie at the tips of the equilateral triangles whose bases are the line connecting the primary masses. These are called the triangular Lagrangian equilibrium points and are stable to small oscillations so long as the mass ratio of the primaries, \( m_2/(m_1 + m_2) < 0.0385 \) (Murray and Dermott 1999). This condition is met for all Sun-planet and planet-moon pairs in the Solar system, with the exception of Pluto and Charon.
The locations of L4 and L5 are easy to find because they form equilateral triangles with the centers of m1 and m2. To locate the other three Lagrange points we can again use Equation 14.

\[
\Omega = \frac{1}{\tilde{r}} + q\left(\frac{1}{\sqrt{\tilde{r}^2 - 2\tilde{r}\lambda + 1}} - \tilde{r}\lambda\right) + \frac{1}{2}(1+q)(1-v^2)\tilde{r}^2
\]

(14)

For L1, \(\theta=90^\circ, \phi=0^\circ\), so \(v=0\) and \(\lambda=1\). So along the positive x-axis,

\[
\Omega_{x+} = \frac{1}{\tilde{r}} + q\left(\frac{1}{\sqrt{\tilde{r}^2 - 2\tilde{r} + 1}} - \tilde{r}\right) + \frac{1}{2}(1+q)\tilde{r}^2.
\]

To find the position of L1, we set the first derivative of this function equal to zero.

\[
g(\tilde{r}) = \frac{d\Omega_{x+}}{d\tilde{r}} = -\frac{1}{\tilde{r}^2} - q\left(\frac{\tilde{r} - 1}{(\tilde{r}^2 - 2\tilde{r} + 1)^{\frac{3}{2}}} + 1\right) + (1+q)\tilde{r} = 0
\]

We can employ the Newton Raphson method to find the solution to this polynomial.

\[
\tilde{r}_{n+1} = \tilde{r}_n - \frac{g(\tilde{r}_n)}{g'(\tilde{r}_n)}
\]

The series converges quickly to find the value of \(\tilde{r}\) for L1 using an initial guess of 0.5. Notice that for L2, the values of \(\theta, \phi, v\) and \(\lambda\) at the same as those for L1. However if we use an initial guess of 1.5 we instead find to find the value of \(\tilde{r}\) for L2.

For L3, \(\theta=90^\circ, \phi=180^\circ\), so \(v=0\) and \(\lambda=-1\). The functions to be used in the Newton Raphson method for L3 are then

\[
\Omega_{x-} = \frac{1}{\tilde{r}} + q\left(\frac{1}{\sqrt{\tilde{r}^2 + 2\tilde{r} + 1}} + \tilde{r}\right) + \frac{1}{2}(1+q)\tilde{r}^2
\]

\[
g(\tilde{r}) = \frac{d\Omega_{x-}}{d\tilde{r}} = -\frac{1}{\tilde{r}^2} - q\left(\frac{\tilde{r} + 1}{(\tilde{r}^2 + 2\tilde{r} + 1)^{\frac{3}{2}}} - 1\right) + (1+q)\tilde{r} = 0
\]

and
\[
g'(\tau) = \frac{d^2\Omega}{d\tau^2} = \frac{2}{\tau^3} - q \left( \frac{1}{(\tau^2 + 2\tau + 1)^{3/2}} - \frac{3\tau}{(\tau^2 + 2\tau + 1)^{3/2}} - \frac{3(\tau + 1)}{(\tau^2 + 2\tau + 1)^{3/2}} \right) + (1 + q)
\]

or more simply

\[
g'(\tau) = \frac{d^2\Omega}{d\tau^2} = \frac{2}{\tau^3} + \frac{2q}{(\tau^2 + 2\tau + 1)^{3/2}} + (1 + q).
\]

**Example:** The table to the right shows the result of the calculations above for a mass ratio of \(q = m_2/m_1 = 0.5\). Note that we can also use the position of \(L_1\) to determine the potential of the Roche Lobes. For \(q=0.5\), we get \(\Omega_{\text{Roche}}=2.876\).

<table>
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<th>(L_i)</th>
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<th>(\tilde{y})</th>
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Excel worksheets, computer programs, and other examples can be found at the website below.

*Dan Bruton, Ph.D.*

*SFA Observatory*

*Department of Physics and Astronomy*

*Stephen F. Austin State University*


**References**

